CERTAIN TOPICS ON THE LINEAR COMPLEMENTARITY PROBLEM

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SHAHNAWAZ AHMED

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Under the guidance of

Dr. K.C. SIVAKUMAR

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY MADRAS

CHENNAI – 600036

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(Shahnawaz Ahmed)

CERTIFICATE

This is certify that **Mr. Shahnawaz Ahmed**, II M.sc. (Mathematics) has worked on the dissertation titled **"Certain Topics on The Linear Complementarity Problem"** under my supervision and guidance.

April 2008

Dr. K.C. SIVAKUMAR

Assistant Professor,
Dept. of Mathematics,
I.I.T Madras,
Chennai – 36.

ABSTRACT

The purpose of this project is to study certain aspects of the linear complementarity problem. We give a comprehensive introduction to the definitions and properties that are relevant to the linear complementarity problem. The principal objective of the present dissertation is to discuss detailed proofs of results on the existence and multiplicity of solutions.

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by

Shahnawaz Ahmed

Department of Mathematics
Indian Institute of Technology Madras
Chennai, India-600036

under the supervision of

Dr. K C Sivakumar

Department of Mathematics
Indian Institute of Technology, Madras
Chennai, India-600036
Email: kcskumar@iitm.ac.in

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1 Introduction

Given a vector

$$q \in \mathbb{R}^n$$

and a matrix

$$M \in \mathbb{R}^{n \times n},$$

the linear complementarity problem denoted LCP(q,M) is to find a vector

$$z \in \mathbb{R}^n$$

such that

$$z \geq 0 \tag{1}$$

$$q + Mz \geq 0 \tag{2}$$

$$z^T(q + Mz) = 0 \tag{3}$$

or to show no such vector z exists. Here, $z \geq 0$ denotes that $z_i \geq 0$ $\forall$ i=1,2,3,...,n, where $z = (z_1, z_2, ..., z_n)$.

A vector z satisfing the inequalities (1) and (2) is said to be feasible. If a feasible vector z satisfies the inequalities in (1) and (2) strictly, then it is called a strictly feasible solution. We say that LCP(q,M) is (strictly) feasible if a (strictly) feasible vector exists. The set of all feasible vectors of LCP(q,M) is called its feasible region and is denoted by FEA (q,M). Let

$$w = q + Mz \tag{4}$$

A feasible vector z of the LCP(q,M) satisfies (3) if and only if

$$z_i w_i = 0 \ \forall \ i = 1, 2, 3, ..., n \tag{5}$$

Condition (5) is often used in place of (3).
The solution set of the Lcp (q,M) is denoted by SOL(q,M). If $q \geq 0$, then LCP(q,M) is always solvable with zero vector being a trivial solution.

Anothe way of representation of the LCP(q,M) is :

$$w \ \geq \ 0, z \ \geq \ 0 \tag{6}$$

$$w = q + Mz \tag{7}$$

$$z^T w = 0 \tag{8}$$

Special Case :

LCP(q,M) where q=0 is called a homogenous LCP. A special property of the LCP(0,M) is that if $z \in SOL(0, M)$, then $\lambda z \in SOL(0, M)$ for all scalars $\lambda \geq 0$. The homogeneous LCP is trivially solved by the zero vector.

2 Useful Terminology

- **Convex Combination :** If $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, then

$$z = \lambda x + (1 - \lambda)y \tag{9}$$

is called a *convex combination* of x and y. More generally, if x^1, x^2, x^3,. ..,$x^k \in \mathbb{R}^n$ and $\lambda_1 + \lambda_2 + ... + \lambda_k = 1$, where $\lambda_1, \lambda_2, ..., \lambda_k$ are nonnegative, then

$$z = \lambda_1 x^1 + ... + \lambda_k x^k \tag{10}$$

is called a *convex combination* of $x^1, x^2, ..., x^k$.

- **Convex Set :** A set C is called convex if and only if it contains any convex combination of any two of its elements.

- **Convex Function :** A real valued function $\theta : \mathfrak{D} \to \mathbb{R}$ defined on the convex set $\mathfrak{D} \subseteq \mathbb{R}^n$ is said to be convex if for any two vectors x and y in $\mathfrak{D}$ and any scalar $\lambda \in [0, 1]$,

$$\theta(\lambda x + (1 - \lambda)y) \leq \lambda\theta(x) + (1 - \lambda)\theta(y) \tag{11}$$

The function θ is said to be strictly convex on $\mathfrak{D}$ if strictly inequality holds for all vectors $x \neq y$ in $\mathfrak{D}$ and $\lambda(0, 1)$.

- **Cone :** We say that a nonempty set X in $\mathbb{R}^n$ is a cone if, for any $x \in X$ and any $t \geq 0$, we have $tx \in X$. If a cone X is, in addition, a convex set, then we say that X is a convex cone. Note that origin is an element of every cone.
A matrix $A \in \mathbb{R}^{m \times p}$ generates a convex cone obtained by taking nonnegative linear combinations of the columns of A. This cone, denoted pos A, is given by

$$pos\ A = \{q \in \mathbb{R}^m : q = Av\ for\ some\ v \in \mathbb{R}^p_+\}, \tag{12}$$

where $\mathbb{R}^p_+ := \{x \in \mathbb{R}^p : x = (x_1, x_2, ..., x_p), x_i > 0\ \forall\ i = 1, 2, ..., p\}$.

- **Principal submatrix :** Let $A \in \mathbb{R}^{m \times n}$ be given. For index set $\alpha \subseteq \{1, 2, ..., m\}$ and $\beta \subseteq \{1, 2, ..., n\}$, the submatrix $A_{\alpha \times \beta}$ of A is the matrix whose entries lie in the rows of A indexed by α and the columns indexed by β. If $\alpha = \{1, 2, ..., m\}$, we denote the submatrix $A_{\alpha\beta}$ by $A_{\bullet\beta}$; similarly, if $\beta = \{1, 2, ..., n\}$, we denote $a_{\alpha\beta}$ by $A_{\alpha\bullet}$. If m=n and $\alpha = \beta$, the submatrix $A_{\alpha\alpha}$ is called a *principal submatrix* of A. The determinant of $A_{\alpha\alpha}$ is called a *principal minor* of A.

- **Leading Principal Submatrix :** Let $A \in \mathbb{R}^{n \times n}$ be given. For a given integer $k(1 \leq k \leq n)$, the principal submatrix $A_{\alpha \times \alpha}$ where $\alpha = \{1, 2, ..., k\}$ is called a *leading principal submarix* of A. The determinant of a leading principal submatrix of A is called a *leading principal minor* of A.

- **Symmetric Matrix :** A matrix $A \in \mathbb{R}^{n \times n}$ is called *symmetric* if $A = A^T$.

- **Positive Definite Matrix :** A matrix $A \in \mathbb{R}^{n \times n}$ is said to be positive semi-definite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. It is called positive definite if $x^T Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

- **Frank-Wolf Theorem :** If the quadratic function f is bounded below on the nonempty polyhedron X, then f attains its infimum on X.[ie, if there exists a real number γ such that $f(x) \geq \gamma$ for all $x \in X$, then there exists a vector $\tilde{x} \in X$ suchthat $f(\tilde{x}) \leq f(x)$ for all $x \in X$]

- **Farkas Lemma :** Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. The system Ax=b, $x \geq 0$ has a solution if and only if the system $y^T A \leq 0, y^T b > 0$ has no solution.

- **Ville's Theorem :** Let $A \in \mathbb{R}^{m \times n}$ be given. The system $Ax > 0, x > 0$ has a solution if and only if the system $y^T A \leq 0, y \geq 0, y \neq 0$ has no solution.

- **Polyhedral Set :** Intersection of planes on a surface.

- **Karush Kuhn Tucker (KKT) Conditions :** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$ be given.
  Consider the problem :

$$\begin{aligned} Min\ z &= f(x) \\ subject\ to\ g(x) &\geq c \\ x &\geq 0 \end{aligned}$$

  To determine stationary points, we introduce a surplus variable and consider the Lagrangian function defined by :

  $L(x, s, \lambda) = f(x) - \lambda(h(x) - s^2)$, where λ is the Lagrangian multiplier.
  A necessary condition for the existance of stationary points are :

$$0 = \frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} \forall\ j = 1, 2, ..., n \tag{13}$$

$$0 = \frac{\partial L}{\partial \lambda} = -[h(x) - s^2] \tag{14}$$

$$0 = \frac{\partial L}{\partial s} = 2\lambda s \tag{15}$$

  The last equation requires either λ=0 or s=0. If s=0 equation (14) implies that h(x)=0. Thus (14) and (15) together imply $\lambda h(x) = 0$.

Thus, a set of necessary conditions for a point x to be a point of minimum are as follows :

$$f_j - \lambda h_j = 0 \;\; \forall \; j = 1, 2, ..., n$$
$$where \; f_j = \frac{\partial f}{\partial x_j} \; and \; h_j = \frac{\partial h}{\partial x_j}$$

$$\lambda h = 0$$
$$h \geq 0$$
$$\lambda \geq 0$$

These conditions are also sufficient if f(x) is convex and h(x) is concave.

3 Quadratic Programming

Quadratic programming (QP) is concerned with the problem of minimizing or maximizing a quadratic function over a polyhedron. Such problems can take many different forms, depending on how the polyhedral feasible region is represented. Let us consider the quadratic programming problem :

$$min\ f(x) \ = c^T x + \frac{1}{2} x^T Q x \tag{16}$$

$$subject\ to\ \ Ax \ \geq b \tag{17}$$

$$x \geq 0 \tag{18}$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

If x is a local optimal solution of the quadratic program then there exists a vector $y \in R^m$ such that the pair (x,y) satisfies the Karush-Kuhn-Tucker conditions :

$$u = c + Qx - A^T y \geq 0,\ x \geq 0,\ x^T u = 0 \tag{19}$$

$$v = -b + Ax \geq 0,\ y \geq 0,\ y^T v = 0 \tag{20}$$

If in addition, Q is positive semi definite, i.e., if the objective function f(x) is convex, then the condition in (19) and (20) are in fact, sufficient for the vector x to be a global optimal solution of the quadratic program.

3.1 Theorem

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a convex function on a convex set $S \subseteq \mathbb{R}^n$. If f has a local minimum on S then this local minimum is also a global minimum on S.

Proof :

Let a local minimum of f be attained at x_0. Then there exists atleast one x_1 in S, $x_1 \neq x_0$ such that $f(x_1) < f(x_0)$. Since f is a convex fuction on S we have $f[\lambda x_1 + (1-\lambda)x_0] \leq \lambda f(x_1) + (1-\lambda) f(x_0)$. Also $\lambda f(x_1) + (1-\lambda)$ $f(x_0) \leq \lambda f(x_0) + (1-\lambda) f(x_0) = f(x_0)$. Thus $f[\lambda x_1 + (1-\lambda)x_0] \leq f(x_0)$. Now for any $\epsilon > 0$, we observe that $|[\lambda x_1 + (1-\lambda)x_0] - x_0| = \lambda |x_1 - x_0| < \epsilon$ so long as $\lambda < \epsilon < |x_1 - x_0|^{-1}$ and such a $\lambda < 1$ does exist whenever $\epsilon < |x_1 - x_0|$. Thus $\lambda x_1 + (1-\lambda)x_0$ will have a smaller value for f(x) in the ϵ-neighbourhood of x_0, whenever λ is so chosen as desired above. This contradicts the fact that f(x) takes on a local minimum at x_0. Hence x_0 is a global minimum.

The Karush-Kuhn-Tucker conditions define the LCP(q,M) where,

$$q = \begin{bmatrix} c \\ -b \end{bmatrix} \; and \; M = \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix}$$

so that

$$q + Mz = \begin{bmatrix} c \\ -b \end{bmatrix} + \begin{bmatrix} Q & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If Q is positive semi definite as in quadratic programming, so is M.

An important special case of the quadratic program is where the only constraints are nonnegetivity restrictions on the variable x. In this case the quadratic program takes the simple form :

$$min \; f(x) = c^T x + \frac{1}{2} x^T Q x \tag{21}$$

$$subject \; to \; x \geq 0$$

If Q is positive semi definite, then the equation defined above is completely equivalent to the LCP(c,Q), where Q is symmetric (by assumption).
If M is asymmetric in the LCP(q,M) then we can associate it with the following quadratic program :

$$min \;\; z^T(q + Mz) \tag{22}$$

$$\begin{aligned} subject \; to \;\; q + Mz &\geq 0 \\ z &\geq 0. \end{aligned}$$

Notice that the objective function of the quadratic program is always bounded below by zero on the feasible set. It is easy to see that a vector that z is a solution of the LCP(q,M) if and only if it is a global minimum of the quadratic program with an objective value of zero.
Previously, we have shown that LCP(q,M) can be represented by the quadratic program :

$$\begin{aligned} min \;\; & z^T(q + Mz) \\ subject \; to \;\; & q + Mz \geq 0 \end{aligned}$$

$$z \geq 0 \tag{23}$$

3.2 Lemma

If LCP(q,M) is feasible, then the quadratic program (23) has an optimal solution $z^\star$. Moreover, there exists a vector $u^\star$ of multipliers satisfying the conditions :

$$q + (M + M^T)z^\star - M^T u^\star \geq 0 \quad (24)$$

$$(z^\star)^T(q + (M + M^T)z^\star - M^T u^\star) = 0 \quad (25)$$

$$u^\star \geq 0 \quad (26)$$

$$(u^\star)^T(q + Mz^\star) = 0 \quad (27)$$

Finally the vectors $z^\star$ and $u^\star$ satisfy

$$(z^\star - u^\star)_i(M^T(z^\star - u^\star))_i \leq 0 \;\forall\; i = 1, 2, ..., n \quad (28)$$

Proof :
Since (q,M) is feasible, so is the quadratic program. As the objective function of the quadratic program is bounded by below on the feasible region, the Frank-Wolfe Theorem implies that there exists an optimal solution for the quadratic program.Such an optimal solution $z^\star$ and a suitable vector $u^\star$ of multipliers will satisfy the Karush-Kuhn Tucker conditions (24) to (27). We have

$$\begin{aligned} & z^T q + z^T Mz \\ &= (q^T z) + \frac{1}{2} z^t (2M) z \\ &= (q^T z) + \frac{1}{2} z^T Mz + \frac{1}{2} z^T Mz \\ &= q^T z + \frac{1}{2} z^T Mz + \frac{1}{2}((Mz)^T z)^T \\ &= q^T z + \frac{1}{2} z^T Mz + \frac{1}{2} z^T M^T z \end{aligned}$$

$$= q^T z + \frac{1}{2} z^T (M + M^T) z \quad (29)$$

Using (29), we can write the quadratic program (23) as :

$$\begin{aligned} min \;\; & q^T z + \frac{1}{2} z^T (M + M^T) z \\ subject\ to\ & q + Mz \geq 0 \\ & z \geq 0 \end{aligned}$$

Here $M + M^T$ is a symmetric matrix.
Applying Karush-Kuhn-Tucker conditions to this quadratic program we get

$$q + (M + M^T)z^\star - M^T u^\star \geq 0 \tag{30}$$
$$(z^\star)^T(q + (M + M^T)z^\star - M^T u^\star) = 0 \tag{31}$$
$$u^\star \geq 0 \tag{32}$$
$$(u^\star)^T(q + Mz^\star) = 0 \tag{33}$$

Equation (31) gives us

$$0 = (z^\star)^T(q + (M + M^T)z^\star - M^T u^\star = (z^\star)^T(q + Mz^\star + M^T(z^\star - u^\star)) = 0$$

As

$$q + Mz^\star \geq 0$$

We conclude that

$$(z^\star)^T(M^T(z^\star - u^\star) \leq 0$$

which at the component level gives

$$(z^\star)_i(M^T(z^\star - u^\star))_i \leq 0. \tag{34}$$

Similarly, multipling the i-th component in (30) by $u_i^\star$ and then invoking the complimentary condition $(u^\star)^T(q + Mz^\star) = 0$ which is implied by (32) and (33), and using the feasiblity of $z^\star$ we get

$$-u_i^\star(M^T(z^\star - u^\star))_i \leq 0.$$

Multipling (30) by $u_i^\star$ and using the fact that $u_i^\star \geq 0$ and appling KKT conditions we get

$$\begin{aligned} 0 &\leq u_i^\star(q)_i + u_i^\star((M + M^T)z^\star) - u_i^\star(M^T u^\star)_i \\ &= u_i^\star(q)_i + u_i^\star(Mz^\star) - u_i^\star(M^T(z^\star - u^\star))_i \\ &= (u^\star)^T(q + Mz^\star) + u_i^\star(M^T(z^\star - u^\star))_i \\ &= u_i^\star(M^T(z^\star - u^\star))_i \end{aligned}$$

Thus,

$$-u_i^\star(M^T(z^\star - u^\star))_i \leq 0 \tag{35}$$

Simply, adding (34) and (35) we get (28) as follows :

$$(z^\star - u^\star)_i(M^T(z^\star - u^\star))_i \leq 0 \ \forall \ i = 1, 2, ..., n \tag{36}$$

3.3 Theorem

Let M be a positive semi definite matrix. If LCP(q,M) is feasible, then it is solvable.

Proof :
By Lemma 3.2 there exists $z^\star$ and $u^\star$ such that $z^\star$ is feasible for LCP(q,M) and conditions (24) to (28) hold. Adding the equalities in (28), we obtain :

$$(z^\star - u^\star)^T M^T (z^\star - u^\star) \leq 0$$

Since M is positive semi-definite,

$$(z^\star - u^\star)^T M^T (z^\star - u^\star) \geq 0$$

Combining these two equations we get

$$(z^\star - u^\star)^T M^T (z^\star - u^\star) = 0,$$

that is,

$$\sum_{i=1}^{n} (z^\star)_i (M^T (z^\star - u^\star))_i + \sum_{i=1}^{n} (-u^\star)_i (M^T (z^\star - u^\star))_i = 0$$

But from (34) and (35) we have

$$(z^\star)_i (M^T (z^\star - u^\star))_i \leq 0 \ \forall \ i = 1, 2, ...n.$$

and

$$(-u^\star)_i (M^T (z^\star - u^\star))_i \leq 0 \ \forall \ i = 1, 2, ...n.$$

Thus

$$(z^\star)_i (M^T (z^\star - u^\star))_i = 0 \ \forall \ i = 1, 2, ...n.$$

Adding all these equations, we get

$$(z^\star)^T (M^T (z^\star - u^\star)) = 0$$

Similarly, we get

$$(-u^\star)^T (M^T (z^\star - u^\star)) = 0$$

From the KKT condition, we have

$$\begin{aligned} 0 &= (z^\star)^T (q + (M + M^T) z^\star - M^T u^\star) \\ &= (z^\star)^T (q + M z^\star + M^T (z^\star - u^\star)) \\ &= (z^\star)^T (q + M z^\star) + (z^\star)^t (M^T (z^\star - u^\star)) \\ &= (z^\star)^T (q + M z^\star) \end{aligned}$$

Thus, $z^\star$ is a solution of LCP(q,M).

3.4 Lemma

If M is a positive definite matrix, then there is a vector $z > 0$ such that $Mz > 0$.

Proof :

Indeed, if no such vector z exists, then by Ville's theorem of the alternative, it follows that there exists a non-zero vector $u \geq 0$ such that $M^T u \leq 0$. Multipling by u yields $u^T M^T u \leq 0$, contradicting the positive definiteness of M.

3.5 Definition

A square matrix M for which a vector $z > 0$ satisfying $Mz > 0$ exists is called an S-matrix (Stiemke).

The class of all S-matrices is denoted by S.

It should be noted that there exists $z > 0$ such that $Mz > 0$ if and only if there exists $z \geq 0$ with $Mz > 0$. Clearly the latter is implied by the former. On the other hand, suppose a vector $z \geq 0$ is given such that $Mz > 0$. Since Mz is continuous in z, it follows that $M(z + \lambda e) \geq 0$ for all $\lambda \geq 0$ small enough. As $z + \lambda e > 0$, we have the former.

Lemma 3.4 shows that a positive definte matrix M must belong to the class S.

3.6 Proposition

The matrix $M \in \mathbb{R}^{n \times n}$ is an S-matrix if and only if LCP(q,M) is feasible for all $q \in \mathbb{R}^n$.

Proof :

Consider an arbitary LCP(q,M) with $M \in S$. Let $\tilde{z} > 0$ be such that $M\tilde{z} > 0$. We have,

$$\begin{aligned} \lambda M\tilde{z} = M(\lambda\tilde{z}) \geq -q \\ \Longrightarrow q + M(\lambda\tilde{z}) \geq 0 \end{aligned}$$

for a suitable large positive scalar λ and since $\lambda\tilde{z} > 0$, we conclude that $\lambda\tilde{z}$ is feasible for (q,M).

Conversely, if (q,M) is feasible for every q, take any $q < 0$. Any feasible solution $\tilde{z}$ of (q,M) will satisfy $M\tilde{z} \geq -q > 0, \tilde{z} \geq 0$. Thus M is an S-matrix.

3.7 Theorem

If $M \in \mathbb{R}^{n \times n}$ is positive definite, then the LCP(q,M) has a unique solution for all $q \in \mathbb{R}^n$.

Proof

Let $M \in \mathbb{R}^{n*n}$ be positive definite. Then $M \in S$. By proposition 3.6, LCP(q,M) is feasible for all $q \in \mathbb{R}^n$. As M is positive definite and the

LCP(q,M) is feasible, it has a solution. As M is positive definite, the objective function is strictly convex. Hence the quadratic program has a unique optimal solution. Consequently so does LCP(q,M).

In general, the LCP with a positive semi definite matrix can have multiple solutions. For instance, the LCP with

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; and \; M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

has solutions $z^1 = (1, 0), z^2 = (0, 1) and z^3 = (\frac{1}{2}, \frac{1}{2})$. Observe that w = q + Mz is the same for all the three solutions $z^i (i = 1, 2, 3)$.

3.8 Theorem

Let $M \in \mathbb{R}^{n \times n}$ is positive definite, and let $q \in \mathbb{R}^n$ be arbitary. The following hold :
(a) If z^1 and z^2 are two solutions of LCP(q,M), then

$$(z^1)^T (q + Mz^2) = (z^2)^T (q + Mz^1) = 0 \tag{37}$$

(b) Suppose that $z^\star \in SOL(q, M)$ has the properties
(i)$z^\star$ is non degenerate.
(ii)$M_{\alpha\alpha}$ is nonsingular where

$$\alpha = \{i : z_i^\star > 0\}$$

Then $z^\star$ is the unique solution of LCP(q,M).

(c) If LCP(q,M) has a solution, the SOL(q,M) is polyhedral and is equal to :

$$P = \{z \in \mathbb{R}_+^n : q + Mz \geq 0, q^T (z - \tilde{z}) = 0, (M + M^T)(z - \tilde{z}) = 0\},$$

where $\tilde{z}$ is an arbitary solution of the LCP(q,M).

(d) If M is symmetric (as well as positive semi-definite), then $Mz^1 = Mz^2$ for any two solutions z^1 and z^2.

Proof :
(a) Let $w^i = q + Mz^i$ for i = 1,2. We have

$$w^1 - w^2 = M(z^1 - z^2)$$

By the positive definiteness of M and by the fact that z^1 and z^2 solve the LCP(q,M), we obtain

$$\begin{aligned} 0 &\leq (z^1 - z^2)^T M (z^1 - z^2) \\ &= ((z^1)^T - (z^2)^T)(Mz^1 - Mz^2) \end{aligned}$$

$$= ((z^1)^T - (z^2)^T)(q + Mz^1 - q - Mz^2)$$
$$= ((z^1)^T - (z^2)^T)(w^1 - w^2)$$
$$= (z^1)^T w^1 - (z^1)^T w^2 - (z^2)^T w^1 + (z^2)^T w^2$$
$$= 0 - (z^1)^T w^2 - (z^2)^T w^1 + 0$$
$$= (z^1)^T w^2 - (z^2)^T w^1$$

$$\leq 0 \tag{38}$$

Consequently we must have $(z^1)^T w^2 = (z^2)^T w^1$=0, as desired, since $z^1, z^2 \geq 0$. This proves (a).

We now demonstrate (b):
Let $\tilde{z}$ be any solution. By (37) we have

$$(q + M\tilde{z})_i = 0 \ \forall \ i \in \alpha \tag{39}$$

If $i \notin \alpha$, then $(q + Mz^\star)_i \geq 0$ by the nondegeneracy of $z^\star$. By (37), we deduce that $\tilde{z}_i = 0 \ for \ i \notin \alpha$. Thus (39) becomes the square system of linear equations $q_\alpha + M_{\alpha\alpha} z_\alpha = 0$, whose solution must be unique by the nonsingular assumption on $M_{\alpha\alpha}$. This proves (b).

Next, we establish (c):
Let $\tilde{z}$ be a given solution and z an arbitary solution. Similar to derivation of (38), it follows that

$$(z - \tilde{z})^T M (z - \tilde{z}) = 0$$

Let $z - \tilde{z} = x$
Then, by using the fact that if a positive semidefinite quadratic form vanishes, then its gradient also vanishes, we have

$$(M + M^T)(z - \tilde{z}) = 0 \tag{40}$$

Thus

$$z^T (M + M^T)(z - \tilde{z}) = 0$$

so that

$$z^T (M + M^T) z = z^T (M + M^T) \tilde{z}.$$

and

$$\tilde{z}^T (M + M^T)(z - \tilde{z}) = 0$$

so that

$$\tilde{z}^T (M + M^T) z = \tilde{z}^T (M + M^T) \tilde{z}.$$

But

$$z^T(M+M^T)\tilde{z} = \tilde{z}^T(M+M^T)z.$$

So,

$$z^T(M+M^T)z = \tilde{z}^T(M+M^T)\tilde{z}.$$

This means,

$$z^TMz + z^TM^Tz = \tilde{z}^TM\tilde{z} + \tilde{z}^TM^T\tilde{z}$$
$$z^TM^Tz = (z^TMz)^T = z^TMz$$

Similarly we can prove the other.
At the same time we have $0 = \tilde{z}^T(q+M\tilde{z}) = z^T(q+Mz)$.Because z and $\tilde{z}$ are soln of the LCP.

Conversely, suppose that $z \in P$.
To show z solves LCP(q,M).
We have,

$$z^T(M+M^T)(z-\tilde{z}) = 0$$

So that

$$z^T(M+M^T)z = z^T(M+M^T)\tilde{z}.$$

Then

$$\tilde{z}^T(M+M^T)z = \tilde{z}^T(M+M^T)\tilde{z}$$

Thus

$$z^T(M+M^T)z = \tilde{z}^T(M+M^T)\tilde{z}$$

So that

$$z^TMz = \tilde{z}^TM\tilde{z}.$$

Also, since

$$q^T(z-\tilde{z}) = 0 \;\; \textit{we have} \;\; z^Tq = \tilde{z}^Tq.$$

Since

$$z^T(q+Mz) = \tilde{z}^T(q+Mz) = 0,$$

We have

$$(M+M^T)(z-\tilde{z}) = 0$$

Thus

$$z^T(q+Mz) = \tilde{z}^T(q+Mz) = 0.$$

Next we show (d):
The hypothesis of this include those of (c). The desired conclusion now follows from the symmetry assumption on M and the condition $(M+M^T)$ $(z-\tilde{z}) = 0$ in the definition of the solution set P.

3.9 Theorem

Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ be given. The following two statements are equivalent:
(a) The solution set of (q, M) is convex.
(b) For any two solutions z^1 and z^2 of (q,M), equation (37) holds.
Moreover, if SOL(q,M) is convex, then it is equal to X_α where
$\alpha = \{i : z_i > 0 \; for \; some \; z \in \; SOL(q, M)\}$.
Proof :$(a) \Longrightarrow (b)$
Let z^1 and z^2 be any two solutions of (q,M). By the convexity assumption, the vector $z = \tau z^1 + (1-\tau)z^2$ is also a solution for any $\tau \in (0, 1)$. By letting $w^i = q + Mz^i$ for i = 1,2, we have
$0 = (\tau w^1 + (1-\tau)w^2)^T(\tau z^1 + (1-\tau z^2)) = \tau(1-\tau)((w^1)^T z^2 + (w^2)^T z^1)$
from which (39) follows.

$(b) \Longrightarrow (a)$:
This follows easily by reversing the argument used in the first part.
Finally, we prove that SOL(q,M), if convex, is equal to X_α. It suffices to show that $SOL(q, M) \subseteq X_\alpha$. But this follows from (37) and the definition of the index set α.

3.10 The Classes Q and $\mathbf{Q_0}$

Definition
The class of matrices M for which the LCP(q,M) has a solution for all vectors q is denoted by **Q** and its elements are called **Q**-matrices.

Definition
The class of matrices M for which the LCP(q,M) is solvable whenever it is feasible is denoted by $\mathbf{Q_0}$ and its elements are called $\mathbf{Q_0}$-matrices.

3.11 Proposition

Let $M \in \mathbb{R}^{n\times n}$. The following are equivalent :
(a) $M \in \mathbf{Q_0}$.
(b) K(M) in convex where K(M) is defined as $K(M) = \{q : SOL(q, M) \neq \phi\}$
(c) K(M) = pos(I,-M)
Proof :$(a) \Longrightarrow (b)$:
Let q^1 and q^2 be two vectors in K(M). Thus, LCP(q^i, M) is solvable for i=1,2. Then, LCP(q^λ, M) is feasible for all $q^\lambda = \lambda q^1 + (1-\lambda)q^2$ with $\lambda \in [0, 1]$. Thus, by (a), it follows that LCP(q^λ, M) is solvable. Hence $q^\lambda \in K(M)$ and (b) follows.
$(b) \Longrightarrow (c)$:
This is clear since the convex hull of K(M) is equal to pos(I,-M).
$(c) \Longrightarrow (a)$:
The cone pos(I,-M) consists of all vectors q for which the LCP(q,M) is

feasible. Therefore if (c) holds, (a) follows.

From the Proposition 3.6, it is clear that the classes $\mathbf{Q}$ and $\mathbf{Q_0}$ are related through the equation :

$$\mathbf{Q} = \mathbf{Q_0} \cap \mathbf{S}$$

4 P-matrices and Global Uniqueness

In this section we establish a characterization for the class of matrices M such that the LCP(q,M) has a unique solution for all vectors q. For this purpose, we introduce the class of **P**-matrices.

4.1 Definition.

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a **P**-matrix if all its principal minors are positive. The class of such matrices is denoted by **P**.

If M is a **P**-matrix, then so are each of its principal submatrices and their transposes. It is well known that a symmetric matrix is positive definite if and only if it belongs to **P**. An example shows us how this equivalence breaks down when the symmetry assumption is dropped. Nevertheless, it will follow directly from a theorem (to be proved later) that any positive definite matrix belongs to the class **P**.

4.2 Example.

Let

$$M = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Thus, M is a **P**-matrix. However, if x = (1,1), we note that $x^T Mx = -1 < 0$, which shows that M is not positive definite.

In order to state the first characterization theorem on **P**-matrices, we introduce the important notion of sign reversal property.

4.3 Definition.

A matrix $M \in R^{n\times n}$ is said to reverse the sign of a vector $z \in \mathbb{R}^n$ if $z_i(Mz_i) \leq 0$ for all i=1,...,n.
Next, we present a characterization of sign reversing matrices.

4.4 Theorem.

Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:
(a) M is a **P**-matrix.
(b) M reverses the sign of no non-zero vector, i.e,

$$[z_i(Mz)_i \leq 0 \ for \ all \ i] \Longrightarrow [z = 0]$$

(c) All real eigenvalues of M and its principal submatrices are positive.

Proof.$(a) \implies (b)$
This is clearly true for n = 1. Using induction, we will now assume this implication holds for the case $n-1$, where $n > 1$. Suppose that the **P**-matrix $M^{n\times n}$ reverses the sign of the non-zero vector $z \in \mathbb{R}^n$. If $z_i = 0$, for some i, then the principal submatrix $M_{\bar{i}\bar{i}}$ is a **P**-matrix which reverses the sign of the non-zero vector $z_{\bar{i}}$. This contradicts the induction hypothesis, so no component of z is zero. We may now write

$$(Mz)_i = d_i z_i \ with\ d_i = (Mz)_i / z_i \leq 0$$

Letting D = diag $(d_1, ..., d_n)$, we obtain (M - D)z = 0. Also we know that

$$det(M - D)z = \sum_{\alpha} det(-D_{\alpha\alpha}) det M_{\bar{\alpha}\bar{\alpha}}$$

where α runs over the index subsets of {1,...,n}. Since D is a nonpositive diagonal matrix and M is a **P**-matrix, it follows that $M - D$ must be nonsingular. Thus, we obtain a contradiction, and (b) follows.

$(b) \implies (c)$
Since M reverses the sign of no non-zero vector, the same can be said of each principal submatrix of M. Hence it suffices to show that all real eigenvalues of M are positive. Let λ be one such eigenvalue and z an associated eigenvector. The vector must be non-zero and, as λ is real, we may take z to be real. We have $Mz = \lambda z$. Since M does not reverse the sign of z, it follows that $\lambda > 0$.

$(c) \implies (a)$
Since the determinant of a matrix is equal to the product of all the (real as well as complex) eigenvalues, and since the complex eigenvalues always appear in conjugate pairs for real matrices, it follows that, if (c) holds, the determinant of M and all its principal submatrices must be positive.

4.5 Corollary

Every **P**-matrix is an **S**-matrix.
Proof :
We will prove the contrapositive. Let $M \in \mathbb{R}^{n\times n}$ be not an **S**-matrix. Then the system $Mz \geq 0$, $z \geq 0$ has no solution for any $z \in \mathbb{R}^n$. Then by Ville's theorem we can say for $v \in \mathbb{R}^n$ the system $v^T M \leq 0, \quad v \geq 0, \quad v \neq 0$ has a solution. It follows that the system $M^T v \leq 0, \quad v \geq 0, \quad v \neq 0$ has a solution. Thus M^T reverses the sign of a non zero vector v. Therefore M^T is not a **P**-matrix. Thus M is not a **P**-matrix.

4.6 Example

The converse part of Corollary 4.5 is false. Let

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Here $v > 0$. We have $Mv = (3,3) > 0$. So M is an **S**-matrix. But M is not a **P**-matrix as its determinant is negative. So we have proved that every **S**-matrix is not a **P**-matrix.

4.7 Theorem

$M \in \mathbb{R}^{n \times n}$ is a **P**-matrix if and only if LCP(q,M) has a unique solution for all vectors $q \in \mathbb{R}^n$.
Proof :
Suppose M is a **P**-matrix, then it follows that it is an **S**-matrix. In particular, by Proposition 3.6, we can say that the LCP(q,M) is feasible for each $q \in \mathbb{R}^n$. By Lemma 3.2 we can say that the quadratic program

$$\begin{aligned} min \;\; & z^T(q + Mz) \\ subject\ to \;\; & q + Mz \geq 0 \\ & z \geq 0 \end{aligned}$$

has an optimal solution $z^\star$. Moreover, there exists a vector $u^\star$ of multipliers satisfying the conditions :

$$q + (M + M^T)z^\star - M^T u^\star \geq 0 \tag{41}$$
$$(z^\star)^T(q + (M + M^T)z^\star - M^T u^\star) = 0 \tag{42}$$
$$u^\star \geq 0 \tag{43}$$
$$(u^\star)^T(q + Mz^\star) = 0 \tag{44}$$
$$(z^\star - u^\star)_i (M^T(z^\star - u^\star))_i \leq 0 \;\forall\; i = 1, 2, ..., n \tag{45}$$

From (45) we see that M^T reverses the sign of $(z^\star - u^\star)$. Now, M is a **P**-matrix and so M^T is also a **P**-matrix. So $(z^\star - u^\star)$ must be the zero vector so that $z^\star = u^\star$. Hence by (44) $z^\star$ solves LCP(q,M).
To show the uniqueness of $z^\star$, let us assume that $\tilde{z}$ is an alternative solution. Write $w^\star = q + Mz^\star \; and \; \tilde{w} = q + M\tilde{z}$. Subtracting we deduce $w^\star - \tilde{w} = M(z^\star - \tilde{z})$. Thus for all i = 1,2,3,...n,

$(z^\star - \tilde{z})_i(w^\star - \tilde{w})_i$
$= (z^\star - \tilde{z})_i(q + w^\star - q - \tilde{w})_i$
$= z_i^\star(q + w^\star)_i - z_i^\star(q + \tilde{w})_i - \tilde{z}_i(q + w^\star)_i + \tilde{z}_i(q + \tilde{w})_i i$
$= -{z^\star}_i(q + \tilde{w})_i - \tilde{z}_i(q + w^\star)_i$
≤ 0

This contrdicts the fact that M reverses the sign of no non-zero vector. Thus $(w^\star - \tilde{w}) = 0$ so that $w^\star = \tilde{w}$.
Conversely suppose that M is not a **P**-matrix. Then there exists a non-zero vector z whose sign is reversed by M. Then $z_i(Mz)_i \leq 0$ for all i. Let $z^+ =$ max (0,z) and $z^- =$ max (0,-z) be the positive and negative parts of z, respectively. As $z \neq 0$ we have $z^+ \neq z^+$. Also we have $z = z^+ - z^-$. Similarly let $u^+ =$ max (0,Mz) and $u^- =$ max (0,-Mz).Also we have $Mz = u^+ - u^-$. We define

$$\tilde{q} = u^+ - Mz^+ = Mz + z^- - M(z + z^-) = u^- - Mz^-.$$

If $z_i \geq 0$, then $(Mz)_i \leq 0$. Thus, $z_i u_i = 0$. Consequently, z^+ is a solution to LCP$(\tilde{q}, M)$. Similarly, we can show that the same is true for z^-. Therefore the LCP$(\tilde{q}, M)$ has two distinct solutions. This completes the proof.

4.8 Definition

$M \in \mathbb{R}^{n*n}$ is said to be (positive) stable if there exists a symmetric positive definite matrix H such that HM is positive definite. M is said to be diagonally (positive) stable if there exists a positive diagonal matrix D such that DM is positive definite.
Clearly, a diagonal stable matrix is in the class P.

4.9 Theorem

Let $M \in \mathbb{R}^{n \times n}$. The following statements are equivalent :
(a) M is diagonally Stable.
(b) There exist positive diagonal matrices D and E such that DME is positive definite.
(c) There exists a positive diagonal matrix F such that $F^{-1}MF$ is positive definite.
Moreover, if M is diagonally stable, then all eigenvalues of M have positive real parts.

Proof
(a) $\Longrightarrow$ (b)
If (a) holds, then DM is positive definite. Choosing E=I, (b) holds.
(b) $\Longrightarrow$ (c)
Let DME be positive definite. Let $(DE)^{\frac{-1}{2}}(DME)(DE)^{\frac{-1}{2}}$ = L. We will prove that L is also positive definite. $x^T Lx = x^T (DE)^{\frac{-1}{2}}(DME)(DE)^{\frac{-1}{2}}x$
Let $(DE)^{\frac{-1}{2}}x = y$. Then $y^T = x^T(DE)^{\frac{-1}{2}}\cdot$ We get $x^T Lx = y^t(DME)y > 0$ $\forall \ x \in \mathbb{R}^n \setminus \{0\}$. So L is positive definite. Let F = $E(DE)^{\frac{-1}{2}}$ and X = $(DE)^{\frac{-1}{2}}$. Then FX = I and so X = F^{-1}. So $F^{-1}MF = (DE)^{\frac{-1}{2}}(DME)(DE)^{\frac{-1}{2}}$ = L. As L is positive definite $F^{-1}MF$ is also positive definite.

(c) $\implies$ (a)
If $F^{-1}MF$ is positive definite, then using the above proof we can say that $F^{-1}(F^{-1}MF)F^{-1} = F^{-2}M$ is also positive definite. As F is a positive diagonal matrix then F^{-2} is also a positive diagonal matrix.
Thus M is diagonally stable.

Finally, let M be a diagonally stable matrix. Then by (c), we may assume that M itself is positive definite. Let $\lambda = a + ib$ be an eigen value of M, and let z = u + iv be a corresponding eigenvector. By equating the real and imaginary parts in the equation Mz = λz, we get Mu = au - bv and Mv = av + bu, from which we deduce $0 < u^T Mu + v^T Mv = a(u^T u + v^T v)$. Thus, $a > 0$.

4.10 Example

Let

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -17 \\ 4 & 0 & 1 \end{bmatrix}$$

The eigenvalues of M are 5 and $-1 \pm i\sqrt{13}$. Thus M can not be scaled positive definite. As all the principal minors of M are positive, M is a **P**-matrix.

4.11 Definition.

A matrix $M \in \mathbb{R}^{n \times n}$ is said to be an **H**-matrix if there exists $d \in \mathbb{R}^n$ with $d > 0$ such that for all i=1,...,n,

$$|m_{ii}|d_i > \sum_{j \neq i} |m_{ij}|d_j.$$

The class of **H**-matrices plays a particularly important role in several areas of the LCP. Among the **H**-matrices, those with positive diagonal entries can be shown to belong to the class **P**; indeed, we will now show that such **H**-matrices are diagonally stable. For this purpose it is useful to introduce the following concept.

4.12 Definition

Let $M \in \mathbb{R}^{n \times n}$. The matrix $\bar{M}$ defined as

$$\bar{m}_{ij} = \begin{cases} |m_{ij}| & for\ i = j \\ -|m_{ij}| & for\ i \neq j \end{cases}$$

is called the comparison matrix associated with M.

Suppose that $\bar{M}$ is an **S**-matrix.
Then there exists a vector $z > 0$ such that

$$\bar{M}z > 0.$$

So,

$$\sum_{j=1}^{n} \bar{m}_{ij} z_j > 0 \quad i = 1, 2, ..., n.$$

and so

$$\bar{m}_{ii} z_i + \sum_{j \neq i} \bar{m}_{ij} z_j > 0$$

i.e.,

$$|m_{ii}| z_i + \sum_{j \neq i} -|m_{ij}| z_j > 0$$

So that

$$|m_{ii}| z_i > \sum_{j \neq i} |m_{ij}| z_j$$

Thus M is a **H**-matrix. In general, if a matrix $\mathbb{M}^{n \times n}$ has positive diagonal entries then for any arbitary vector $z \in \mathbb{R}^n$

$$z^T M z \geq |z|^T \bar{M} |z|$$

Thus, if $\bar{M}$ is positive definite (positive semi-definite), then so is M.

4.13 Lemma

Let $M \in \mathbb{R}^{n \times n}$. If $\bar{M} \in \mathbf{S}$, then $\bar{M} \in \mathbf{P}$.

Proof: As $\bar{M}$ is in **S**, there is a $d > 0$ for which $\bar{M}d > 0$. Thus, taking D = diag (d), it follows that $\bar{M}D$ is a strictly row diagonally dominant. As $\bar{M}D$ is an **S**-matrix, M is a **H**-matrix.
Thus

$$|m_{ii}| d_i > \sum_{j \neq i} |m_{ij}| d_j$$

So that

$$|\bar{M}_{ii}| d_i > \sum_{j \neq i} |\bar{m}_{ij}| d_j$$

Thus

$$|\bar{M}_{ii}d_i| > \sum_{j\neq i} |\bar{m}_{ij}d_j|$$

and so $\bar{M}D$ is strictly row diagonally dominant.
Let $z \in \mathbb{R}^n$ be a arbitary non-zero vector. We may assume z_1 is the component with the largest absolute value. Let us assume that z_1 is positive. Then

$$\begin{aligned} z_1(\bar{M}z)_1 &= z_1(|\bar{m}_{11}|d_1z_1 - \sum_{j\neq 1} |\bar{m}_{1j}|d_jz_j) \\ &> z_1(\sum_{j=1}^{n} |\bar{m}_{1j}|d_j(z_1 - z_j)) \\ &> 0 \end{aligned}$$

If z_1 is negative then similarly It can be shown that $z_1(\bar{M}z)_1$ is positive. Thus, $\bar{M}D$ reverses the sign of no non-zero vector. Therefore by the Theorem 4.4, $\bar{M}D$ is in **P**, hence, so is $\bar{M}$.

4.14 Theorem

Let $M \in \mathbb{R}^{n\times n}$. If M is a **H**-matrix with positive diagonal entries, then M is diagonally stable.

Proof. Let $\bar{M}$ be the comparison matrix of M. As M is an **H**-matrix, then $\bar{M}$ is an **S**-matrix. Then we can find a positive diagonal matrix D such that $\bar{M}D$ is strictly row dominant. i.e, $\bar{M}De > 0$. From Lemma 4.13, we know that $\bar{M}D$ is a **P**-matrix. Thus $D\bar{M}^T$ is also a **P**-matrix as the transpose of a **P**-matrix is again a **P**-matrix. It follows from Corollary 4.5, that $D\bar{M}^T$ is also an **S**-matrix as every **P**-matrix is a **S**-matrix. Again, this means that a positive diagonal matrix E exists such that $N = D\bar{M}^TE$ is strictly row diagonally dominant. Notice that N is also strictly column diagonally dominant as $N^T = E\bar{M}D$ is strictly row diagonally dominant. Thus $(N + N^T)$ is symmetric, equal to its own comparison matrix, and strictly row and column diagonally dominant. From Lemma 4.4, $(N + N^t)$ is in **P** and hence is positive definite. Therfore N is positive definite and from $z^tMz \geq |z|^t\bar{M}|z|$ we see that EMD is positive definite. Theorem 4.9, now implies that M is diagonally stable.

5 $\mathbf{P_0}$-matrices and w-Uniqueness

In Theorem 3.8(d), we showed that if M is a symmetric positive semi-definite matrix, then any two solutions z^1 and z^2 of the LCP(q,M) give rise to the same vector $w = q + Mz^i$ (i = 1,2). This property of w-uniqueness can be characterized by a certain condition on M related to the notion of sign reversing. The characterization is somewhat like that of z-uniqueness by means of the **P**-property. Before establishing criteria for w-uniqueness, we define a generalization of the class **P**.

5.1 Definition

$M \in \mathbb{R}^{n\times n}$ is said to be a $\mathbf{P_0}$-matrix if all its principal minors are nonnegative. The class of all such matrices is denoted by $\mathbf{P_0}$.

Parallel to the fact that any positive definite matrix must belong to **P**, it will follow that a positive semi-definite matrix must belong to P_0. The following result gives some characterizations for a $\mathbf{P_0}$-matrix very much like those for a **P**-matrix.

5.2 Theorem

Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:
(a) M is a $\mathbf{P_0}$-matrix.
(b) For each vector $z \neq 0$, there exists an index k such that $z_k \neq 0$ and $z_k(Mz)_k \geq 0$.
(c) All real eigen values of M and its principal submatrices are nonnegative.
(d) For each $\epsilon > 0$, $M + \epsilon I$ is a **P**-matrix.

Proof. $(a) \Longrightarrow (d)$
For any diagonal matrix D,

$$det(M - D)z = \sum_{\alpha} det(-D_{\alpha\alpha}) det M_{\bar{\alpha}\bar{\alpha}}$$

where α runs over the index set subsets of {1,...,n}. Let $D = \epsilon I$ in the above expression. As M is a $\mathbf{P_0}$-matrix, each term in the sum will be nonnegative and the term with $\alpha = \{1, ..., n\}$ will be $\epsilon^n > 0$. Thus, det$(M + \epsilon I) > 0$. As each principal submatrix of M must also be in $\mathbf{P_0}$, this argument applies to each of these submatrices. Therefore, $M + \epsilon I$ is a **P**-matrix.

$(d) \Longrightarrow (b)$
Let $z \neq 0$ be given. Since $M + \epsilon I$ is a **P**-matrix for $\epsilon > 0$, there esixts an index set i (depending on ϵ) such that $z_i((M + \epsilon I)z)_i > 0$. Let $\{\epsilon_k\}$ be a sequence converging to zero. There must exist an index j such that $z_j((M + \epsilon_k I)z)_j > 0$ for infinitely many ϵ_k. Clearly $z_j \neq 0$. Also, as $k \longrightarrow \infty$, we have $\epsilon_k \longrightarrow 0$, and we deduce that $z_j(Mz)_j \geq 0$, as desired.

$(b) \Longrightarrow (c)$
Since M reverses the sign of no non-zero vector, the same can be said of each principal submatrix of M. Hence it suffices to show that all real eigenvalues of M are positive. Let λ be one such eigenvalue and z an associated eigenvector. The vector must be non-zero and, as λ is real, we may take z to be real. We have $Mz = \lambda z$. Since M does not reverse the sign of z, it follows that $\lambda > 0$.

$(c) \Longrightarrow (a)$
Since the determinant of a matrix is equal to the product of all the (real as well as complex) eigenvalues, and since the complex eigenvalues always appear in conjugate pairs for real matrices, it follows that, if (c) holds, the determinant of M and all its principal submatrices must be positive.

5.3 Example

The condition that $z_k \neq 0$ in Theorem 5.2(b), is essential. Let us consider the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is clearly not in $\mathbf{P_0}$. Given any $z = (z_1, z_2) \neq 0$, we find

$$z_1(Mz)_1 = z_1^2 \geq 0.$$

Thus, M would satisfy 5.2(b) if we dropped the condition that $z_k \neq 0$. It can be seen that the matrix M does not meet the full requirement of Theorem 5.2(b) by letting z=(0,1).

5.4 Theorem

Let $M \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

(a) For all $q \in K(M)$, if z_1 and z_2 are any two solutions of LCP(q,M), then $Mz^1 = Mz^2$.
(b) Every vector whose sign is reversed by M belongs to the null space of M, i.e,

$$[z_i(Mz)_i \leq 0 \;\; for \; all \; i = 1, ..., n] \Longrightarrow [Mz = 0] \tag{46}$$

(c) M is a $\mathbf{P_0}$-matirx and for each index set α with det$M_{\alpha\alpha}$=0, the columns of $M_{\bullet\alpha}$ are linearly dependent.

Proof.
$(a) \implies (b)$.
Suppose there exists a non-zero vector z such that $z_i(Mz)_i \leq 0$ for all i and $Mz \neq 0$. Let z^+=max(0,z) and z^-=max(0,-z) be the positive and negative parts of z, respectively. As $z \neq 0$, we have $z^+ \neq z^-$. Similarly, let u^+=max(0,Mz) and u^-=max(0,-Mz). Noticing that $z = z^+ + z^-$ and $Mz = u^+ + u^-$, we define

$$\bar{q} = u^+ - Mz^+ = u^- - mz^-.$$

If $z_i > 0$, then $(Mz)_i \leq 0$. Thus, $z_i^+ u_i^+ = 0$. Consequently, z^+ is a solution of $(\bar{q}, M)$. Similarly, one can show that the same is true for z^-. Therefore LCP$\bar{q}, M$ has two distinct solutions z^+ *and* z^- with

$$\bar{q} + Mz^+ = u^+ \neq u^- = \bar{q} + Mz^-,$$

which contradicts (a).

$(b) \implies (c)$.
If the implication (46) holds, it follows from Theorem 5.2(b) that M must be a $\mathbf{P_0}$-matrix. Suppose that there is an index set α for which det$M_{\alpha\alpha} = 0$. Then there is a vector $z_\alpha \neq 0$ such that $M_{\alpha\alpha} z_\alpha = 0$. Define $z = (z_\alpha, 0)$. The non-zero vector z satisfies $[z_i(Mz)_i] = 0 \; for \; all \; i$. By (48), we must have $0 = Mz = M_{\bullet\alpha} z_\alpha$. Hence the columns of $M_{\bullet\alpha}$ are linearly dependent, and (c) follows.

$(c) \implies (b)$.
Let $\hat{z}$ be a vector whose sign is reversed by M. Without loss of generality, we may assume that $\hat{z}$ is nonnegative. (Otherwise, we may apply the argument below to the matrix $\bar{M} = DMD$ where D is the diagonal matrix with entries $d_{ii} = 1$ if $\hat{z}_i \geq 0$ and $d_{ii} = -1$ if $\hat{z}_i < 0$. The matrix $\bar{M}$ is easily seen to satisfy the assumptions of (c), given that M does. Moreover, $\bar{z}_i(\bar{M}\bar{z})_i \leq 0$ for all i=1,...,n where $\bar{z} = D\hat{z} \geq 0$). Let $\hat{w} = M\hat{z}$. Suppose that $\hat{w} \neq 0$. The system

$$\hat{w} = Mz, z \geq 0$$

has a solution, $\hat{z}$. Let α be the support of $\hat{z}$, thus $\alpha \neq \phi$. Notice, $\hat{w}_\alpha \leq 0$ as $\hat{z}_\alpha > 0$. By linear programming theory, there is a (basic) feasible solution $\tilde{z} \geq 0$ with support $\beta \subseteq \alpha$ such that $M_{\bullet\beta}$ has linearly independent columns. Note, $\beta \neq \phi$ as $0 \neq \hat{w} = M\tilde{z}$. Obviously, $M_{\beta\beta}$ is a $\mathbf{P_0}$-matrix. As a matter of fact, $M_{\beta\beta}$ is a $\mathbf{P}$-matrix for if det $M_{\gamma\gamma} = 0$ for some $\gamma \subseteq \beta$, then the columns of $M_{\bullet\gamma}$ are linearly dependent, contradicting the linear independence of $M_{\bullet\alpha}$. As $\beta \subseteq \alpha$, we find that $M_{\beta\beta}$ reverses the sign of the positive vector $\tilde{z}_\beta$. This is a contradiction and, therefore (b) follows.

$(b) \implies (a)$.

To show z^1 is the unique solution, let z^2 be an alternative solution. Let $w^1 = q + Mz^1$ *and* $w^2 = q + Mz^2$. Subtracting, we deduce that $w^1 - w^2 = M(z^1 - z^2)$. Thus, for all i=1,...,n,

$$0 \geq (z^1 - z^2)_i(w^1 - w^2)_i = (z^1 - z^2)_i(M(z^1 - z^2))_i,$$

contradicting the fact that M reverses the sign of no non-zero vector. This implies$M(z_1 - z_2 = 0)$ and (a) is proved.

5.5 Remark

It is clear that a nonsingular matrix satisfying the condition Theorem 5.4(c) must be a **P**-matrix.

5.6 Definition

A matrix $M \in \mathbf{P_0} \cap \mathbb{R}^{n \times n}$ is said to be
(a) column adequate if for each $\alpha \subseteq \{1, ..., n\}$

$$[det M_{\alpha\alpha} = 0] \Longrightarrow [M_{\bullet\alpha} \textit{ has linearly dipendent columns}].$$

(b) row adequate if M^T is column adequate.

(c) adequate if M is both column and row adequate.

5.7 Remark.

According to the theorem 5.4, the column adequacy of M characterizes the uniqueness of the w-part of any solution of the LCP(q,M) for all vectors q.

Obviously, any **P**-matrix is adequate, as is any symmetric positive semi-definite matrix. Not every (asymmetric) positive semi-definite matrix is adequate, however. Moreover there are adequate matrices which are neither in the class **P** nor positive semi-definite.

5.8 Example.

The matrix

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

is positive semi-definite, but is neither row nor column adequate. The matrix

$$M = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

is adequate but is neither a **P**-matrix nor is positive semi definite.

Reference

[1] Richard W. Cottle, Jong-Shi Pang, Richard E. Stone, The Linear Complementarity Problem, Academic Press Limited, 1992.